

Constructing minimal blocking sets using field reduction

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Abstract

We present a construction for minimal blocking sets with respect to $(k-1)$ -spaces in $\text{PG}(n-1, q^t)$, the $(n-1)$ -dimensional projective space over the finite field \mathbb{F}_{q^t} of order q^t . The construction relies on the use of blocking cones in the *field reduced* representation of $\text{PG}(n-1, q^t)$, extending the well-known construction of linear blocking sets. This construction is inspired by the construction for minimal blocking sets with respect to the hyperplanes by Mazzocca, Polverino and Storme (*the MPS-construction*); we show that for a suitable choice of the blocking cone over a planar blocking set, we obtain larger blocking sets than the ones obtained from planar blocking sets in [15].

Furthermore we show that every minimal blocking set with respect to the hyperplanes in $\text{PG}(n-1, q^t)$ can be obtained by applying field reduction to a minimal blocking set with respect to $(nt-t-1)$ -spaces in $\text{PG}(nt-1, q)$. We end by relating these constructions to the linearity conjecture for small minimal blocking sets. We show that if a small minimal blocking set is constructed from the MPS-construction, it is of Rédei-type whereas a small minimal blocking set arises from our cone construction if and only if it is linear.

Keywords: field reduction, blocking set, Desarguesian spread, linear set, linearity conjecture

1 Introduction

This paper is inspired by the paper [16], where Mazzocca, Polverino and Storme construct minimal blocking sets with respect to the hyperplanes in $\text{PG}(n, q^t)$ by using certain cones in the Barlotti-Cofman representation of $\text{PG}(n, q^t)$, extending the results of [15] to general dimension. Our paper is organised as follows. In Section 2, we give the necessary background on the Barlotti-Cofman and field reduced representation of $\text{PG}(n, q)$ and recall the correspondence between these representations. In Section 3, we explain how the construction of Mazzocca, Polverino and Storme (also called the *MPS-construction*) can be presented in an easier way by making use of field reduction: the obtained blocking set corresponds to the points of a minimal blocking set (with respect to subspaces of a

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particular dimension) in a projective space over \mathbb{F}_q , considered over \mathbb{F}_{q^t} . We also show that the construction of linear blocking sets and the recent construction by Costa [6] fit in this framework. By using cones in the field reduced representation of $\text{PG}(n, q^t)$ we generalise the MPS-construction in Section 4. Starting from a planar blocking set, we construct non-planar blocking sets with respect to $(k-1)$ -spaces in $\text{PG}(n-1, q^t)$. In Corollary 4.11, we show that if we choose the defining blocking cone carefully, we construct blocking sets whose size exceeds the ones obtained from the MPS-construction using planar blocking sets.

Finally, in Section 5, we show that *every* minimal blocking set with respect to the hyperplanes in $\text{PG}(n-1, q^t)$ can be obtained by applying field reduction to a minimal blocking set with respect to $(nt-t-1)$ -spaces in $\text{PG}(nt-1, q)$. This also provides us with a different view on the linearity conjecture for small minimal blocking sets. Finally, we show that if a small minimal blocking set is obtained by the MPS-construction then it is of Rédei-type, whereas a small minimal blocking set arises from our Construction 1 if and only if it is a linear blocking set.

2 The Barlotti-Cofman representation and field reduction

2.1 Desarguesian spreads and field reduction

Throughout this paper, we let $\text{PG}(m-1, q)$ denote the $(m-1)$ -dimensional projective space over the finite field \mathbb{F}_q of order q . A $(t-1)$ -spread of a projective space $\text{PG}(m-1, q)$ is a family of mutually disjoint subspaces of dimension $(t-1)$ partitioning the space $\text{PG}(m-1, q)$. It is not hard to show that if a $(t-1)$ -spread of $\text{PG}(m-1, q)$ exists, then t divides m . On the other hand, if t divides m , there exists a $(t-1)$ -spread of $\text{PG}(m-1, q)$. This was already shown by Segre [18], and can also be seen as follows.

By *field reduction* every point of $\text{PG}(n-1, q^t)$ corresponds to a 1-dimensional vector space over \mathbb{F}_{q^t} , which is a t -dimensional vector space over \mathbb{F}_q , and hence, also corresponds to a projective $(t-1)$ -dimensional space over \mathbb{F}_q . The set of all $(t-1)$ -spaces obtained in this way forms a spread of $\text{PG}(nt-1, q)$, which is called a *Desarguesian* $(t-1)$ -spread. Throughout this paper, this $(t-1)$ -spread in $\text{PG}(nt-1, q)$ is fixed and is denoted by \mathcal{D} . A \mathcal{D} -subspace of $\text{PG}(nt-1, q)$ is a space spanned by elements of \mathcal{D} . It follows from the construction that a \mathcal{D} -subspace is partitioned by elements of \mathcal{D} and corresponds to a field reduced subspace π of $\text{PG}(n-1, q^t)$. If the subspace π has dimension $r-1$, then we say that the \mathcal{D} -subspace of dimension $rt-1$ corresponding to π is a \mathcal{D}_{r-1} -subspace.

The following statement is well-known and can be proven by a straightforward counting argument. It will be of use later in this paper.

Lemma 2.1. *Every hyperplane of $\text{PG}(nt-1, q)$ contains exactly one \mathcal{D}_{n-2} -subspace, i.e. an $(nt-t-1)$ -space spanned by elements of \mathcal{D} .*

If U is a subset of $\text{PG}(nt-1, q)$, then we define $\mathcal{B}(U) := \{R \in \mathcal{D} \mid U \cap R \neq \emptyset\}$. In this paper, we identify the elements of $\mathcal{B}(U)$ with their corresponding points of $\text{PG}(n-1, q^t)$. Linear sets can be defined in several equivalent ways, but using the terminology of this paper, an \mathbb{F}_q -linear set S in $\text{PG}(n-1, q^t)$ is a set of points such that $S = \mathcal{B}(\mu)$, where μ

is a subspace of $\text{PG}(nt - 1, q)$. For more information on field reduction and linear sets, we refer to [12].

2.2 The Barlotti-Cofman representation

Let H be a hyperplane of $\text{PG}(n - 1, q^t)$; by field reduction H corresponds to a \mathcal{D}_{n-2} -subspace Σ of $\text{PG}(nt - 1, q)$. Note that Σ has dimension $nt - t - 1$. Let Σ' be an $(nt - t)$ -space through Σ in $\text{PG}(nt - 1, q)$.

Consider the following geometry $\Pi_{n-1} = \Pi_{n-1}(\Sigma', \Sigma, \mathcal{S})$, where \mathcal{S} is the set of elements of \mathcal{D} contained in Σ :

- Points: the points of $\Sigma' \setminus \Sigma$ and the elements of \mathcal{S} .
- Lines: the t -subspaces of Σ' meeting Σ exactly in an element of \mathcal{S} , together with the \mathcal{D}_1 -subspaces contained in Σ .
- Incidence: natural.

The incidence structure Π_{n-1} is isomorphic to the design obtained by taking points and lines of $\text{PG}(n - 1, q^t)$ and we say that Π_{n-1} is the *Barlotti-Cofman representation* of $\text{PG}(n - 1, q^t)$ [1].

2.3 The correspondence between the Barlotti-Cofman representation and the representation using field reduction

From the definitions, we get that the geometry \mathcal{G} with as points the spread elements of \mathcal{D} and as lines the \mathcal{D}_1 -spaces of $\text{PG}(nt - 1, q)$ is isomorphic to the design of points and lines of $\text{PG}(n - 1, q^t)$. Let, as in the previous section, Σ' be an $(nt - t)$ -space through the \mathcal{D}_{n-2} -space Σ in $\text{PG}(nt - 1, q)$.

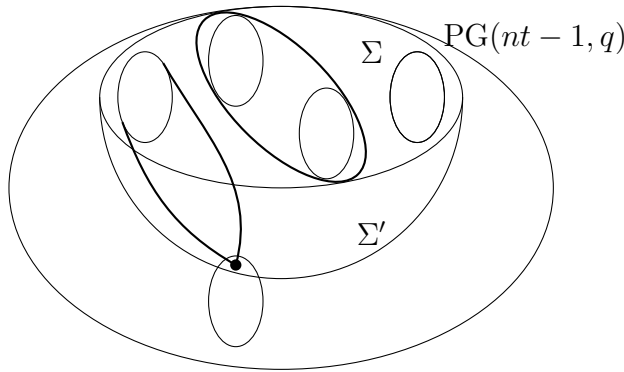


Figure 1: The Barlotti-Cofman representation inside the field reduction representation

It is clear that the Barlotti-Cofman representation Π_{n-1} and \mathcal{G} are isomorphic: consider the following mapping ϕ

$$\begin{aligned}
\phi : \mathcal{G} &\rightarrow \Pi_{n-1} \\
R \not\subset \Sigma &\mapsto R \cap \Sigma' \text{ for } R \in \mathcal{D} \\
R \subset \Sigma &\mapsto R \text{ for } R \in \mathcal{D} \\
L \not\subset \Sigma &\mapsto L \cap \Sigma', \text{ for } L \text{ a } \mathcal{D}_1\text{-space} \\
L \subset \Sigma &\mapsto L, \text{ for } L \text{ a } \mathcal{D}_1\text{-space.}
\end{aligned}$$

It is easy to see that ϕ defines an isomorphism between \mathcal{G} and Π_{n-1} . This isomorphism will enable us to describe the MPS-construction in an easier way.

3 Blocking sets

A *blocking set* B in $\text{PG}(n, q)$ with respect to k -spaces is a set of points such that every k -dimensional space in $\text{PG}(n, q)$ (or k -space) contains at least one point of B . We also say that the set B *blocks* all k -spaces. If we are considering blocking sets with respect to the hyperplanes, we simply say that B is a blocking set. A *minimal* blocking set B (w.r.t. k -spaces) is a blocking set such that no proper subset of B is a blocking set (w.r.t. k -spaces). An *essential point* of a blocking set with respect to k -spaces B is a point lying on a tangent k -space to B and we see that B is minimal if and only if every point of B is essential. A blocking set w.r.t. k -spaces in $\text{PG}(n, q)$ is called *trivial* if it contains an $(n - k)$ -space. A *small* blocking set in $\text{PG}(n, q)$ with respect to k -spaces is a blocking set of size smaller than $3(q^{n-k} + 1)/2$. A blocking set B (w.r.t. hyperplanes) in $\text{PG}(n, q)$ is of *Rédei-type* if there exists a hyperplane meeting B in $|B| - q$ points.

Most constructions for blocking sets concern the planar case or the case of blocking sets with respect to the hyperplanes in $\text{PG}(n, q)$. For blocking sets with respect to k -spaces, $k \neq n - 1$, there are results of Beutelspacher [2] and Heim [8] classifying the smallest non-trivial blocking sets as cones with base a blocking set in a plane. Other results characterise the smallest blocking sets that span a space of a certain fixed dimension [4] or aim at classifying small minimal blocking sets as *linear sets* (see Section 5).

3.1 The cone construction for blocking sets

We recall that the *cone* K with *vertex* Ω , where Ω is a subspace of $\text{PG}(n, q)$ and *base* \bar{B} , contained in a subspace Γ , skew from Ω , is the set $\cup_{\bar{P} \in \bar{B}} \langle \bar{P}, \Omega \rangle$.

The following lemma is well-known, but since we did not find an exact reference, we give a proof for completeness.

Lemma 3.1. *Let Ω be an s -dimensional subspace of $\text{PG}(n, q)$, let Γ be an $(n - s - 1)$ -space disjoint from Ω . The set \bar{B} is a minimal blocking set with respect to k -spaces of the space $\Gamma = \text{PG}(n - s - 1, q)$ ($k < n - s - 1$), if and only if the cone K with vertex Ω and base \bar{B} is a minimal blocking set with respect to k -spaces of $\text{PG}(n, q)$.*

Proof. First assume that \bar{B} is a minimal blocking set with respect to k -spaces of Γ . Let μ be a k -space of $\text{PG}(n, q)$. If μ meets Ω , then K blocks Ω , so assume that μ is skew from Ω . The projection of μ from Ω onto Γ (i.e. $\langle \Omega, \mu \rangle \cap \Gamma$) is a k -space μ' , which contains

at least one point P of \bar{B} . This implies that μ meets $\langle \Omega, P \rangle$ in at least one point, hence, μ contains a point of K . This shows that K is a blocking set. We will now show that K is minimal. Since \bar{B} is a minimal blocking set with respect to k -spaces in Γ , every point Q of \bar{B} lies on a tangent k -space T_Q . Let ν be a $(k-1)$ -space in T_Q , not through Q , then for every point R of Ω , $\langle R, \nu \rangle$ is a tangent k -space to K through the point R , so every point of Ω is essential. Now let S be a point of K , not in Ω . The projection of S onto Γ is a point S' of \bar{B} , lying on a tangent k -space $T_{S'}$. The space $\langle \Omega, T_{S'} \rangle$ is $(k+s+1)$ -dimensional, hence, we can take a k -dimensional subspace of $\langle \Omega, T_{S'} \rangle$, meeting $\langle \Omega, S' \rangle$ in only the point S , which is a tangent k -space through S to K .

Conversely, if K is a minimal blocking set, every k -space in $\text{PG}(n, q)$, and hence in Γ is blocked by the points of K , hence, \bar{B} is a blocking set with respect to k -spaces. Since K is minimal, there exists a tangent k -space T_P to K through every point P of \bar{B} . It is clear that the projection of T_P from Ω onto Γ is a tangent k -space through P to \bar{B} , hence, \bar{B} is minimal. \square

Remark 1. *The cone K with vertex Ω , where Ω is an s -dimensional subspace of $\text{PG}(n, q)$, and base \bar{B} (contained in an $(n-s-1)$ -space Γ , skew from Ω) has size $q^{s+1}|\bar{B}| + \frac{q^{s+1}-1}{q-1}$.*

3.2 Linear blocking sets and the MPS-construction

The following lemma is essentially a trivial observation, but it is the key idea behind the constructions presented in this paper.

Lemma 3.2. *Let B' be a blocking set with respect to $(kt-1)$ -spaces in $\text{PG}(nt-1, q)$, then $B = \mathcal{B}(B')$ is a blocking set with respect to $(k-1)$ -spaces in $\text{PG}(n-1, q^t)$.*

Proof. As B' blocks all $(kt-1)$ -spaces in $\text{PG}(nt-1, q)$, it also blocks the $(kt-1)$ -spaces spanned by spread elements (i.e. the \mathcal{D}_{k-1} -spaces), which means that all $(k-1)$ -spaces of $\text{PG}(n-1, q^t)$ contain at least one point of $\mathcal{B}(B')$. \square

The previous lemma provides us with a way of creating blocking sets B in $\text{PG}(n-1, q^t)$, using blocking sets B' in $\text{PG}(nt-1, q)$. An important problem is to determine when the obtained blocking set B is minimal. In the MPS-construction and the construction of Costa, particular minimal blocking sets B' are considered to ensure that $\mathcal{B}(B')$ is minimal. In the next subsections, we recall these constructions and translate them from the Barlotti-Cofman setting to the setting using field reduction which enables an easier description. Since *linear* blocking sets also fit in this framework and will be used later in this paper, we discuss them here.

3.2.1 Linear blocking sets

Linear blocking sets with respect to $(k-1)$ -spaces in $\text{PG}(n-1, q^t)$ were introduced by Lunardon [13]: he argues that an \mathbb{F}_q -linear set of rank $nt - kt + 1$ is a blocking set. In view of Lemma 3.2, this is clear: we take B' to be an $(nt - kt)$ -dimensional subspace of $\text{PG}(nt-1, q)$, which is a blocking set with respect to $(kt-1)$ -spaces, to obtain a (linear) blocking set $\mathcal{B}(B')$.

The importance of this construction lies in the fact that it provided counterexamples to the belief that all small minimal blocking set were of Rédei-type [17]. We will come back to this in Section 5.2.

It is important to note that linear blocking sets are necessarily minimal blocking sets. This was shown in [14] for \mathbb{F}_q -linear blocking sets with respect to lines in $\text{PG}(n-1, q^t)$. But for blocking sets with respect to $(k-1)$ -spaces in $\text{PG}(n-1, q^t)$, $k \neq 2$, the minimality is up to our knowledge not directly proven in the literature; it does however follow easily from the following lemma of Szőnyi and Weiner.

Lemma 3.3. [21, Lemma 3.1] *Let B be a blocking set of $\text{PG}(n-1, q)$ with respect to $(k-1)$ -dimensional subspaces, $q = p^h$, p prime, and suppose that $|B| \leq 2q^{n-k}$. Assume that each $(k-1)$ -dimensional subspace of $\text{PG}(n, q)$ intersects B in $1 \bmod p$ points. Then B is minimal.*

Since every subspace meets an \mathbb{F}_q -linear blocking set in $1 \bmod q$ points, and an \mathbb{F}_q -linear blocking set with respect to $(k-1)$ -spaces in $\text{PG}(n-1, q^t)$ has size at most $(q^{nt-kt+1} - 1)/(q - 1)$, this implies the following.

Corollary 3.4. *An \mathbb{F}_q -linear blocking set with respect to $(k-1)$ -spaces in $\text{PG}(n-1, q^t)$ is minimal.*

The fact that linear blocking sets are minimal will follow directly from Theorem 4.4.

3.2.2 The MPS-construction

The MPS-construction goes as follows. Let \mathcal{S} be a Desarguesian $(t-1)$ -spread of $\Sigma = \text{PG}(nt-t-1, q)$, embed Σ as a hyperplane in $\Sigma' = \text{PG}(nt-t, q)$ and consider the Barlotti-Cofman representation of $\text{PG}(n-1, q^t)$ as $\Pi_{n-1}(\Sigma', \Sigma, \mathcal{S})$ as described in Subsection 2. Let Y be a fixed element of \mathcal{S} and let Ω be a hyperplane of Y . Let Γ' be an $(nt-2t+1)$ -dimensional subspace of Σ' , disjoint from Ω and denote by Γ the $(nt-2t)$ -dimensional subspace intersection of Γ' and Σ and by T the intersection point of Γ and Y . Let \bar{B} be a blocking set of Γ' such that $\bar{B} \cap \Gamma = \{Q\}$, Q a point, with the property that $\ell \setminus \{T\} \not\subset \bar{B}$, for every line ℓ of Γ' through T . Denote by K the cone with vertex Ω and base \bar{B} . Let B be the subset of Π_{n-1} defined by

$$B = (K \setminus \Sigma) \cup \{X \in \mathcal{S} : X \cap K \neq \emptyset\}.$$

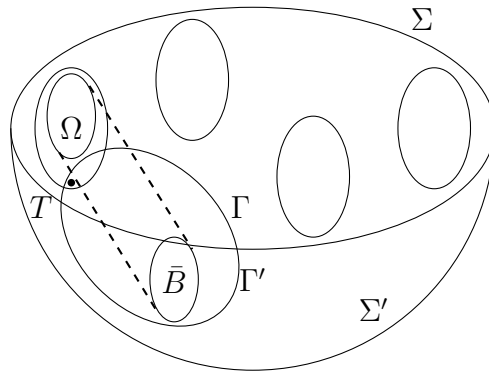


Figure 2: The MPS-construction

With these definitions, the authors show:

Theorem 3.5. [16, Proposition 1] *The set B is a blocking set of the projective space $\text{PG}(n-1, q^t)$.*

Let B be the blocking set obtained by the MPS-construction, then, by using the correspondence between the Barlotti-Cofman-representation and the representation using field reduction (see Subsection 2), we see that B corresponds to $\mathcal{B}(K)$ by using the above definitions. We also see that the set K is a blocking set with respect to $(nt-t-1)$ -spaces in $\text{PG}(nt-1, q)$: by Lemma 3.1, K is a blocking set w.r.t. $(nt-2t)$ -spaces in $\langle \Omega, \Gamma' \rangle$. Clearly every $(nt-t-1)$ -space of $\text{PG}(nt-1, q)$ meets $\langle \Omega, \Gamma' \rangle$ in a space of dimension at least $nt-2t$.

The authors distinguish two different cases of their construction: the case $T = Q$ (which they call Construction A) and the case $T \neq Q$ (Construction B).

The authors show for both cases:

Theorem 3.6. [16, Proposition 2] *B is a minimal blocking set of $\text{PG}(n-1, q^t)$ if and only if \bar{B} is a minimal blocking set of Γ' .*

Remark 2. *The construction of Costa [6] generalises Construction A as follows.*

The notations $\mathcal{S}, \Sigma, \Sigma'$ and Y are used as before. Let Ω be an s -dimensional subspace of Y , with $0 \leq s \leq t-2$, let Γ' be an $(nt-t-s-1)$ -dimensional subspace of Σ skew from Ω , let Γ be the $(nt-t-s-2)$ -dimensional intersection of Γ' and Σ , let θ be the $(t-s-2)$ -space $Y \cap \Gamma$. Define R to be the set $\{\langle S, \Omega \rangle \cap \Gamma' : S \text{ is a hyperplane of } \Pi_{n-1}, \text{ not containing } Y\}$. Let \bar{B} be a blocking set with respect to elements of the set R , containing the space θ . In the same way as before, B is defined to be the set of points of the cone with vertex Ω and base \bar{B} , contained in $\Sigma' \setminus \Sigma$, together with the point Y .

Costa shows that B is a minimal blocking set in $\Pi_{n-1} = \text{PG}(n-1, q^t)$ if and only if \bar{B} is a minimal blocking set with respect to elements of R .

Remark 3. *The MPS-construction and the construction of Costa generalise the cone construction of Lemma 3.1 in some sense. Instead of considering the cone with base a blocking set over \mathbb{F}_{q^t} , these constructions use cones over blocking sets over a subfield \mathbb{F}_{q^t} , i.e., a line through a point of the vertex and the base is no longer a full line, but a subline. The same idea was used for planar blocking sets before in [15] and [20].*

4 Constructing minimal blocking sets with respect to $(k-1)$ -spaces in $\text{PG}(n-1, q^t)$

4.1 A general construction method

Construction 1. Let Ω be an $(nt-kt-2)$ -dimensional subspace of $\text{PG}(nt-1, q)$, let \bar{B} be a blocking set, contained in a plane Γ which is skew from Ω and let K be the cone in $\Pi = \langle \Omega, \Gamma \rangle$ with vertex Ω and base \bar{B} . Let $B = \mathcal{B}(K)$.

Lemma 4.1. *The set B of Construction 1 is a blocking set with respect to $(k-1)$ -spaces in $\text{PG}(n-1, q^t)$.*

Proof. By Lemma 3.1, the cone with base \bar{B} and vertex Ω is a blocking set with respect to lines in $\langle \Omega, \Gamma \rangle$, hence, with respect to $(kt-1)$ -spaces in $\text{PG}(nt-1, q)$. So the statement follows from Lemma 3.2. \square

Lemma 4.2. *Let Π be an $(nt - kt + 1)$ -dimensional subspace of $\text{PG}(nt - 1, q)$, $k \geq 2$. Let \mathcal{D} be a Desarguesian $(t - 1)$ -spread in $\text{PG}(nt - 1, q)$ and let P be a point of Ω . Then there exists a \mathcal{D}_{k-2} -space meeting Ω only in points of $\mathcal{B}(P)$. Moreover, if $\dim(\mathcal{B}(P) \cap \Omega) \geq 1$, i.e. if the spread element through P meets Ω in a subspace of dimension at least 1, then there is a \mathcal{D}_{k-1} -space meeting Ω only in points of $\mathcal{B}(P)$.*

Proof. We will first show that there exists a \mathcal{D}_s -space containing only points of $\mathcal{B}(P)$ of Π , where $0 \leq s \leq k - 2$. The statement holds trivially for $s = 0$. Suppose that the statement holds for some $0 < i \leq k - 3$ and let D be the obtained \mathcal{D}_i -space. The number of \mathcal{D}_{i+1} -spaces through D is $\frac{(q^t)^{n-i-1}-1}{q^t-1}$. If $\mathcal{B}(P)$ meets Π in a space of dimension r , then the number of \mathcal{D}_{i+1} -spaces through D that meet Π in a point, not belonging to $\mathcal{B}(P)$ is at most $\frac{q^{nt-kt-r+1}-1}{q-1}$, since different \mathcal{D}_{i+1} -spaces through D that meet Π meet in spaces that have only the points of $\mathcal{B}(P)$ in common. Now $\frac{q^{nt-kt-r+1}-1}{q-1} < \frac{(q^t)^{n-i-1}-1}{q^t-1}$ for all $r \geq 0$, and $i \leq k - 3$, hence, at least one of the \mathcal{D}_{i+1} -spaces through D meets Π only in points of $\mathcal{B}(P)$. By induction we find a \mathcal{D}_{k-2} -space T containing only points of $\mathcal{B}(P)$ of Π .

If $\mathcal{B}(P)$ meets Π in a space of dimension at least one, then the above count with $r \geq 1$ and $i = k - 2$ shows that there exists a \mathcal{D}_{k-1} -space through T meeting Π only in $\mathcal{B}(P)$. \square

Lemma 4.3. *Let Ω be an $(nt - kt - 2)$ -dimensional subspace of $\text{PG}(nt - 1, q)$, let \bar{B} be a minimal blocking set, contained in the plane Γ which is skew from Ω . Let K be the cone in $\Pi = \langle \Omega, \Gamma \rangle$ with vertex Ω and base \bar{B} . Suppose that every point of \bar{B} lies on at least t tangent lines to \bar{B} in Γ . If P is a point of K , then P lies in at least t hyperplanes H_P of Π that meet K only in some fixed subspace μ of Π of codimension 2.*

Proof. Let P be a point of K , not in Ω , and let P' be the point $\langle \Omega, P \rangle \cap \Gamma$, which is contained in \bar{B} . By assumption, there are at least t tangent lines $\ell_i^{P'}$, $i = 1, \dots, t, \dots$ through P' to \bar{B} . The space $\langle \ell_i^{P'}, \Omega \rangle$ is a hyperplane of Π which meets K only in the space $\langle \Omega, P \rangle$ which has codimension 2 in Π .

Let P be a point of Ω . By choosing all hyperplanes $\langle \Omega, \ell_i^Q \rangle$ with Q arbitrary in \bar{B} we satisfy the required condition. \square

Theorem 4.4. *If \bar{B} is a minimal blocking set in Γ such that every point of \bar{B} lies on at least 2 tangent lines to \bar{B} , then the blocking set $B = \mathcal{B}(K)$ obtained from Construction 1 is minimal.*

Proof. Let P be a point of K . We need to show that there is a \mathcal{D}_{k-1} -space through P containing only points of $\mathcal{B}(P)$ of K . If $\mathcal{B}(P)$ meets Π in a space of dimension at least one, then by Lemma 4.2, there exists a \mathcal{D}_{k-1} -space meeting Π and hence K only in $\mathcal{B}(P)$, which implies that there is a tangent $(k - 1)$ -space through $\mathcal{B}(P)$ to B in $\text{PG}(n - 1, q^t)$.

So from now on, we suppose that $\mathcal{B}(P)$ meets Π only in the point P . By Lemma 4.2, we have that there exists a \mathcal{D}_{k-2} -space T meeting Π only in the point P . By Lemma 4.3, there are (at least) two hyperplanes, say H_1 and H_2 , of Π through P which meet K only in a subspace μ of Π of codimension 2.

Consider the quotient Π/T in the quotient space $\text{PG}(nt - 1, q)/T \cong \text{PG}(nt - kt + t - 1, q)$. Note that, as T is spanned by spread elements of the Desarguesian spread \mathcal{D} , \mathcal{D} induces a Desarguesian $(t - 1)$ -spread \mathcal{D}' in $\text{PG}(nt - 1, q)/T$. The \mathcal{D}_{k-1} -spaces through

T are in one-to-one correspondence with the elements of \mathcal{D}' . Since a tangent \mathcal{D}_{k-1} -space through P to K corresponds to an element of \mathcal{D}' which meets Π/T in a subspace skew to K/T , we need to show that there exists an element of \mathcal{D}' meeting Π/T in a subspace skew to K/T . Note that Π/T is $(nt - kt)$ -dimensional and that H_1/T and H_2/T are hyperplanes of Π/T through μ/T .

Let A be the number of elements of \mathcal{D}' meeting Π/T in a point, then expressing that $A + (\frac{q^{nt-kt+t}-1}{q^t-1} - A)(q+1)$ is at most $\frac{q^{nt-kt+1}-1}{q-1}$, the number of points in Π/T , yields that A is at least $\frac{q+1}{q}(\frac{q^{nt-kt+t}-1}{q^t-1}) - \frac{q^{nt-kt+1}-1}{q(q-1)}$. This implies that there are at most $\frac{q^{nt-kt+1}-1}{q-1} - A < 2q^{nt-kt-1}$ points of Π/T that are not the exact intersection of an element of \mathcal{D}' with Π/T . This implies that there is at least one point of H_1/T or H_2/T that induces a tangent \mathcal{D}_{k-1} -space through T which in turn implies that we have found a tangent $(k-1)$ -space to B in the point $\mathcal{B}(P)$ of $\text{PG}(n-1, q^t)$ and that P is essential. \square

Remark 4. By [5, 9], an affine blocking set contains at least $2q-1$ points. This implies that every point of a minimal blocking set \bar{B} in $\text{PG}(2, q)$ with $|\bar{B}| = q+k, k \leq q$, lies on at least $q-k+1$ tangent lines to \bar{B} . So every minimal blocking set of size at most $2q-1$ satisfies the condition of Construction 1. In particular, if \bar{B} is a line, we find that, as announced before, a linear blocking set is minimal.

Using that the blocking set constructed by Construction 1 is contained in $\mathcal{B}(\Pi)$, where Π is an $(nt - kt + 1)$ -dimensional subspace of $\text{PG}(nt-1, q)$, the following corollary easily follows.

Corollary 4.5. *There exists a minimal $(k-1)$ -blocking set in $\text{PG}(n-1, q^t)$ constructed by Construction 1 which spans an s -dimensional space for all $n-k \leq s \leq \min\{n-1, nt-kt+1\}$.*

Remark 5. If we compare Theorem 4.4 to Theorem 3.6 of Mazzocca, Polverino and Storme, we see that in the latter theorem the vice versa part also holds: if B is minimal, then \bar{B} is necessarily minimal. In general this does not hold for our construction.

4.2 Construction 1 in a scattered subspace

In general, since the position of the space $\Pi = \langle \Omega, \Gamma \rangle$ with respect to the Desarguesian spread \mathcal{D} is arbitrary, we cannot derive the size of the blocking set B from the size of the blocking set \bar{B} . But if we take the space Π to be e.g. a *scattered* subspace with respect to \mathcal{D} (i.e. every element of \mathcal{D} that meets this subspace, meets it in exactly 1 point), we are able to derive the size of B in terms of $|\bar{B}|$. Note that this is a restriction: it is clear that not all examples arising from Construction 1 can be obtained from a scattered subspace Π . In some cases, it is even impossible to find a scattered subspace of the right dimension, in view of the following theorem:

Theorem 4.6. [3, Theorem 4.3] *A scattered \mathbb{F}_q -linear set in $\text{PG}(n-1, q^t)$ has rank $\leq nt/2$.*

In [10, Theorem 2.5.5], Lavrauw shows the following:

Theorem 4.7. *If r is even, then there exists a scattered subspace (with respect to a Desarguesian $(t-1)$ -spread) of dimension $rt/2 - 1$ in $\text{PG}(rt-1, q)$. If r is odd, there exists a scattered subspace (with respect to a Desarguesian $(t-1)$ -spread) of dimension $(rt-t)/2 - 1$.*

Theorem 4.8. *Let $k \geq (n+3)/2$. If \bar{B} is a minimal blocking set in $\text{PG}(2, q)$ such that every point of \bar{B} lies on at least 2 tangent lines to \bar{B} , then there exists a minimal blocking set with respect to $(k-1)$ -spaces in $\text{PG}(n-1, q^t)$ with size $|\bar{B}|q^{nt-kt-1} + \frac{q^{nt-kt-1}-1}{q-1}$.*

Proof. Consider an $(nt-kt+1)$ -dimensional space $\Pi = \langle \Omega, \Gamma \rangle$, where Ω is $(nt-kt-1)$ -dimensional and Γ a plane skew from Ω and such that Π is scattered with respect to \mathcal{D} , which is possible in view of Theorem 4.7 since $k \geq (n+3)/2$. Apply Construction 1 with the spaces Ω, Γ and the minimal blocking set \bar{B} in Γ . Then, by Theorem 4.4, $\mathcal{B}(K)$ is a minimal blocking set with respect to $(k-1)$ -spaces. Moreover, since every element of \mathcal{D} that meets $\langle \Omega, \Gamma \rangle$, meets it in exactly one point, the number of points in B is equal to the number of points in K , which is equal to $|\bar{B}|q^{nt-kt-1} + \frac{q^{nt-kt-1}-1}{q-1}$. \square

The following corollary shows which are the possible dimensions spanned by a blocking set obtained by Construction 1 in a scattered subspace, again using Theorem 4.7.

Corollary 4.9. *If \bar{B} is a minimal blocking set in $\text{PG}(2, q)$ such that every point of \bar{B} lies on at least 2 tangent lines to \bar{B} and if $nt-kt+1 \leq \frac{n't-t}{2} - 1$ and $n'-1 \leq nt-kt+1$, then there is a $(k-1)$ -blocking set B in $\text{PG}(n-1, q^t)$ spanning an $(n'-1)$ -space, where B has size $|\bar{B}|q^{nt-kt-1} + \frac{q^{nt-kt-1}-1}{q-1}$.*

4.3 A modification of Construction 1

A disadvantage of Construction 1 is clearly the requirement that every point lies on at least two tangent lines to \bar{B} . Using the computer package ‘FinInG’ for GAP [7], we found that we cannot remove this condition in general: there are $(t+1)$ -subspaces in $\text{PG}(3t-1, q)$ such that a cone over a Hermitian curve does not define a minimal blocking set in $\text{PG}(2, q^t)$. In this subsection, we choose a particular subspace of $\text{PG}(nt-1, q)$ so that we do not need at least two tangent lines to \bar{B} in order to have minimality.

Construction 2. Let $t \geq 4$. Let ν be a \mathcal{D}_{n-k-1} -space in $\text{PG}(nt-1, q)$ and let Π be an $(nt-kt+1)$ -space through ν such that $\dim(\langle \mathcal{B}(\Pi) \rangle) = n-k+1$. Let Ω be an $(nt-kt-2)$ -dimensional subspace of Π , meeting ν in an $(nt-kt-4)$ -dimensional space, let \bar{B} be a minimal blocking set, contained in the plane Γ of Π , which is skew from Ω and such that \bar{B} is skew from ν . Let K be the cone with vertex Ω and base \bar{B} and let $B = \mathcal{B}(K)$.

Theorem 4.10. *The set B from Construction 2 is a minimal $(k-1)$ -blocking set in $\text{PG}(n-1, q^t)$.*

Proof. The fact that B is a $(k-1)$ -blocking set follows from Lemma 3.2.

The set $\mathcal{B}(\Pi)$ in $\text{PG}(n-1, q^t)$ is a cone C with vertex an $(n-k-1)$ -space $\mathcal{B}(\nu)$ and base an \mathbb{F}_q -subline ℓ . In the quotient space $\text{PG}(n-1, q^t)/\mathcal{B}(\nu)$ we easily find a

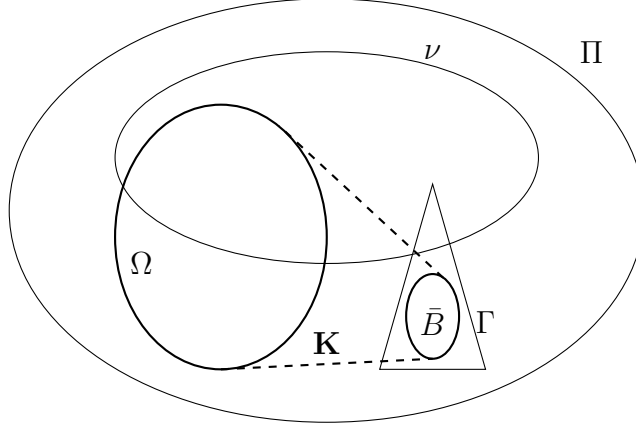


Figure 3: Construction 2

$(k-2)$ -space skew from $\ell/\mathcal{B}(\nu)$, which forces the existence of a $(k-2)$ -space S skew from C . Every point P in $K \cap \nu$ corresponds to a point $\mathcal{B}(P)$ of B lying on the tangent $(k-1)$ -space $\langle \mathcal{B}(\mu), S \rangle$, which means that the point $\mathcal{B}(P)$ is essential if $P \in \nu \cap K$.

Now let P be a point of K , not contained in ν . As in the proof of Theorem 4.4, we may restrict ourselves to the case where $\mathcal{B}(P) \cap \Pi = \{P\}$ and we know that there exists a \mathcal{D}_{k-2} -space T through $\mathcal{B}(P)$ such that T meets Π only in the point P . We have that the quotient space Π/T in $\text{PG}(nt-1, q)/T \cong \text{PG}(nt-kt+t-1, q)$ is $(nt-kt)$ -dimensional and that \mathcal{D} induces a Desarguesian spread \mathcal{D}' in $\text{PG}(nt-kt+t-1, q)$. Since Π/T contains a \mathcal{D}_{n-k-1} -space ν/T , every element of \mathcal{D}' not meeting ν/T meets Π/T in a single point. By Lemma 4.3, since \bar{B} is minimal, P lies on at least one hyperplane H of Π meeting K only in a space of codimension 2.

Since Ω meets ν in an $(nt-kt-4)$ -space, the $(nt-kt-1)$ -space H/T , is different from the $(nt-kt-1)$ -space ν/T . So we can consider a point Q of H/T in Π/T which is not contained in K nor ν/T . This point Q corresponds to a spread element \mathcal{D}' which together with T induces a tangent \mathcal{D}_{k-1} -space to $\mathcal{B}(P)$ in $\text{PG}(n-1, q^t)$, which shows that P is essential. \square

Note that by construction, $\mathcal{B}(K)$ contains an $(n-k)$ -space.

Proposition 1. The $(k-1)$ -blocking set B in $\text{PG}(n-1, q^t)$ obtained from Construction 2 spans a space of dimension $(n-k+1)$.

Proof. The set B is contained in the subspace $\langle \mathcal{B}(\Pi) \rangle$, which spans an $(n-k+1)$ -dimensional space. Note that it is by construction not possible that $\langle \mathcal{B}(\Pi) \rangle = n-k+1$ and $\langle K \rangle = n-k$. \square

Proposition 2. Let Π be so that $\langle \mathcal{B}(\Pi) \rangle$ is $(n-k+1)$ -dimensional. The $(k-1)$ -blocking set B of $\text{PG}(n-1, q^t)$ has size

$$|\bar{B}|(q^{nt-kt-1} - q^{nt-kt-3}) + q^{nt-kt-2} + q^{nt-kt-3} + \epsilon,$$

where ϵ is at least 1.

Proof. Every spread element of \mathcal{D} meeting Π is either entirely contained in Π or meets Π in a point. The cone K consists of a union of $(nt - kt - 1)$ -spaces through Ω . Each of these $(nt - kt - 1)$ -spaces contains $q^{nt-kt-1}$ points, not in Ω of which, since \bar{B} does not meet ν , $q^{nt-kt-1} - q^{nt-kt-3}$ are not contained in ν . There are $q^{nt-kt-2} + q^{nt-kt-3}$ points in $\Omega \setminus \nu$. The only points of B that we have not counted yet are the points of K in ν . It is clear that, since Ω meets ν non-trivially, this number is at least 1. \square

Applying the previous proposition for $k = n - 1$, and using that a \mathcal{D}_{n-k-1} -space is a spread element, we find the following corollary. The second part follows from the fact that a Hermitian curve in $\text{PG}(2, q)$, q square, is a minimal blocking set of size $q\sqrt{q} + 1$.

Corollary 4.11. *The blocking set B in $\text{PG}(n - 1, q^t)$ obtained from a planar blocking set \bar{B} by Construction 2 has size*

$$|\bar{B}|(q^{t-1} - q^{t-3}) + q^{t-2} + q^{t-3} + 1.$$

In particular, if q is a square, we find minimal blocking sets in $\text{PG}(2, q^t)$ of size $q^t\sqrt{q} + q^{t-1} - q^{t-2}\sqrt{q} + q^{t-2} + 1$.

In [15], Mazzocca and Polverino construct blocking sets in $\text{PG}(2, q^t)$ arising from cones in $\text{PG}(2t, q)$. We want to point out that, starting from a Hermitian curve \bar{B} in $\text{PG}(2, q)$, q square, they construct minimal blocking sets in $\text{PG}(2, q^t)$ of size $q^{t-1}(|\bar{B}| - 1) + 1 = q^t\sqrt{q} + 1$, which is smaller than the size of the minimal blocking set obtained from a Hermitian curve in the previous corollary.

5 Minimal blocking sets with respect to the hyperplanes

5.1 The main observation

We have seen in Lemma 3.2 that, if B' is a blocking set with respect to $(nt - t - 1)$ -spaces in $\text{PG}(nt - 1, q)$, then $B = \mathcal{B}(B')$ is a blocking set (w.r.t. hyperplanes) in $\text{PG}(n - 1, q^t)$. In the following theorem, we show that a kind of reverse statement also holds.

Theorem 5.1. *Let B be a minimal blocking set with respect to the hyperplanes of $\text{PG}(n - 1, q^t)$, then B can be written as $\mathcal{B}(B')$, where B' is a minimal blocking set with respect to $(nt - t - 1)$ -spaces of $\text{PG}(nt - 1, q)$.*

Proof. Let $\mathcal{S}(B)$ denote the set of spread elements corresponding to the points of B and let \tilde{B} be the point set of the elements of $\mathcal{S}(B)$. Since B is a blocking set with respect to the hyperplanes of $\text{PG}(n - 1, q^t)$, every \mathcal{D}_{n-2} -space contains an element of $\mathcal{S}(B)$, and hence, certainly a point of \tilde{B} (in fact, at least $\frac{q^t-1}{q-1}$ of them).

Now consider an $(nt - t - 1)$ -space π of $\text{PG}(nt - 1, q)$ which is not a \mathcal{D}_{n-2} -space. Let H be a hyperplane of $\text{PG}(nt - 1, q)$ through π . We know from Lemma 2.1 that H contains a \mathcal{D}_{n-2} -space π' . Since B is a blocking set with respect to the hyperplanes, the space π' contains at least one element of $\mathcal{S}(B)$, say S . Now $\pi \cap \pi'$ is at least $(nt - 2t)$ -dimensional, hence, since S is $(t - 1)$ -dimensional and contained in π' , S meets π non-trivially, and

hence, π contains at least one point of the set \tilde{B} . This implies that \tilde{B} is a blocking set with respect to $(nt - t - 1)$ -spaces.

Now let B' be a minimal blocking set with respect to $(nt - t - 1)$ -spaces contained in \tilde{B} (B' is not necessarily unique). To show that $\mathcal{B}(B') = B$, we show that in every element of $\mathcal{S}(B)$, there lies at least one point of B' . Suppose that there is an element of $\mathcal{S}(B)$, say T , that does not contain a point of B' . Since B is a minimal blocking set, there is a tangent hyperplane in $\text{PG}(n - 1, q^t)$ to B in the point corresponding to T . This tangent hyperplane corresponds to a \mathcal{D}_{n-2} -space (which is an $(nt - t - 1)$ -space) meeting $\mathcal{S}(B)$ in exactly the element T , hence, does not contain a point of the blocking set with respect to $(nt - t - 1)$ -spaces B' , a contradiction. \square

Remark 6. *The blocking set B' found in the previous theorem is not necessarily unique. Consider for example the blocking set B in $\text{PG}(2, q^t)$ consisting of all points of a line. Then B corresponds to the set of spread elements in a \mathcal{D}_1 -space π of $\text{PG}(3t - 1, q)$. It is clear that $B = \mathcal{B}(\mu)$ for any subspace μ of dimension t in π . Take a plane ν in π and a $(2t - 4)$ -dimensional subspace ν' in π , skew from ν , then the cone with vertex ν' and base a large minimal blocking set in ν (e.g. a Hermitian curve if q is a square) is a minimal blocking set B'' with respect to $(t - 1)$ -spaces in π , hence, $B = \mathcal{B}(B'')$. Note that μ is a small minimal blocking set with respect to $(2t - 1)$ -spaces in $\text{PG}(3t - 1, q)$, whereas B'' is a large minimal blocking set with respect to $(2t - 1)$ -spaces in $\text{PG}(3t - 1, q)$.*

Remark 7. *Let B be a Hermitian curve in $\text{PG}(2, q^2)$. Then B is a minimal blocking set of size $q^3 + 1$. If we apply field reduction to B , then we obtain a set $\mathcal{S}(B)$ of $q^3 + 1$ lines in $\text{PG}(5, q)$ of which the point set \tilde{B} blocks all 3-spaces by Theorem 5.1. The $(q^3 + 1)(q + 1)$ points do not form a minimal blocking set with respect to 3-spaces. It is not too hard to check that \tilde{B} is the point set of an elliptic quadric in $\text{PG}(5, q)$ (see e.g. [12]) and that the $q^3 + q^2 + q + 1$ points of a parabolic quadric $\mathcal{Q}(4, q)$ in \tilde{B} form a minimal blocking set B' with respect to 3-spaces in $\text{PG}(5, q)$, hence, $B = \mathcal{B}(B')$. But in general, it is not easy to construct a minimal blocking set B' with respect to $(nt - t - 1)$ -spaces contained in the point set of the spread elements corresponding to B .*

5.2 Small minimal blocking sets and the linearity conjecture

The *linearity conjecture* for small blocking sets states that all small minimal blocking sets in $\text{PG}(n - 1, q^t)$ are *linear sets* over \mathbb{F}_p , where $q^t = p^h$, p prime (see [22]).

We have the following result of T. Szőnyi and Zs. Weiner [21], showing that the linearity conjecture holds for projective spaces over fields of prime order.

Theorem 5.2. *[21, Corollary 3.15] A small minimal blocking set with respect to k -spaces in $\text{PG}(n, p)$, p prime, is an $(n - k)$ -space.*

In general, the linearity conjecture for small minimal blocking sets remains open. For an overview of the cases in which the linearity conjecture has proven to be true, we refer to [12].

If B is a small minimal blocking set with respect to the hyperplanes of $\text{PG}(n - 1, p^h)$, p prime, by Theorem 5.1, B can be written as $\mathcal{B}(B')$ where B' is a minimal blocking set with respect to $(hn - h - 1)$ -spaces of $\text{PG}(nh - 1, p)$. If this latter blocking set is small then by Theorem 5.2, it is an h -space. Hence, the linearity conjecture can be restated as

follows (compare to the statement of Theorem 5.1):

Linearity conjecture: Let B be a small minimal blocking set with respect to the hyperplanes of $\text{PG}(n-1, p^h)$, p prime, then B can be written as $\mathcal{B}(B')$, where B' is a *small* minimal blocking set with respect to $(nh - h - 1)$ -spaces of $\text{PG}(nh - 1, p)$.

We will now investigate the properties of small minimal blocking sets that arise from the MPS-construction and from Construction 1; we will show that the obtained blocking sets are linear blocking sets.

Define the *exponent* e of a small minimal blocking set B with respect to k -spaces as the largest integer such that every k -space meets B in $1 \bmod p^e$ points. Lemma 5.3(1) will show that e is well-defined.

We will need the following properties of small minimal blocking sets in $\text{PG}(n, p^t)$, p prime. For item (2), for simplicity, we state a slightly weaker bound than the one proven in [21, Theorem 3.9].

- Lemma 5.3.**
1. [21, Proposition 2.7] Every k -space meets a small minimal blocking set with respect to k -spaces in $\text{PG}(n, p^t)$, p prime, in $1 \bmod p$ points, hence, $e \geq 1$.
 2. [21, Theorem 3.9] A small minimal blocking set in $\text{PG}(n, p^t)$, $p \geq 7$, with exponent e has at most $p^t + 2p^{t-e} + 1$ points.
 3. [21, Corollary 3.11] Every subspace meets a small minimal blocking set with respect to k -spaces with exponent e in $1 \bmod p^e$ or zero points.
 4. [23, Lemma 6] Let B be a small minimal blocking set with exponent e in $\text{PG}(n, p^t)$, p prime. If for a certain line L , $|L \cap B| = p^e + 1$, then \mathbb{F}_{p^e} is a subfield of \mathbb{F}_{p^t} and $L \cap B$ is \mathbb{F}_{p^e} -linear.
 5. [23, Lemma 4] A point of a small minimal blocking set B with exponent e in $\text{PG}(n, p^t)$, $p \geq 7$, p prime, lying on a $(p^e + 1)$ -secant, lies on at least $p^{t-e} - 4p^{t-e-1} + 1$ $(p^e + 1)$ -secants.

From Lemma 5.3(3) we easily deduce the following corollary which also reproves Theorem 5.2.

Corollary 5.4. A small minimal blocking set with respect to k -spaces with exponent t in $\text{PG}(n, p^t)$, p prime, is an $(n - k)$ -space.

Theorem 5.5. If a small minimal blocking set B in $\text{PG}(n-1, p^t)$, p prime, arises from the MPS construction in $\text{PG}(nt-1, p)$ then it is linear and of Rédei-type.

Proof. We use the notations of Section 3.2.2. Let B be a small minimal blocking set with respect to the hyperplanes in $\text{PG}(n-1, p^t)$ arising from the MPS-construction, then $B = \mathcal{B}(K)$, where K is a blocking set with respect to the hyperplanes of $\Sigma' = \text{PG}(nt-t, p)$ and K meets Σ in a $(t-1)$ -space. Since all points of B , not in the hyperplane corresponding to the \mathcal{D}_{n-2} -space Σ correspond to a unique point of K , and $|B| \leq p^t + p^{t-1} + 1$ we have that K has at most $p^t + 2p^{t-1} + \frac{p^t-1}{p-1}$ points, which implies that K is small. Since K is a minimal blocking set by Theorem 3.6, it is a t -space by Theorem 5.2.

It follows that B is \mathbb{F}_p -linear. If the $(t-1)$ -space $K \cap \Sigma$ is not contained in the spread element Y , then Σ corresponds to a hyperplane containing $p^{t-1} + 1 = |B| - p^t$ points of B , hence, B is of Rédei-type. If $K \cap \Sigma$ equals Y , then let Q be a point of K , not in Σ . It is clear that, since K is a t -space through the spread element Y , K is contained in the \mathcal{D}_1 -space $\langle Y, \mathcal{B}(Y) \rangle$, hence B is a line. \square

Remark 8. *The vice versa part of the previous theorem does not hold: let π be a t -space in $\text{PG}(nt-1, p)$ such that there is an element Y of \mathcal{D} meeting π in a $(t-3)$ -space and an element Z meeting π in a line and such that π is not entirely contained in $\langle Y, Z \rangle$. Then the line corresponding to the \mathcal{D}_1 -space $\langle Y, Z \rangle$ clearly contains $|B| - p^t$ points of $\mathcal{B}(\pi)$, hence, the small linear blocking set $\mathcal{B}(\pi)$ is of Rédei-type, but there does not exist an element of \mathcal{D} meeting π in a $(t-2)$ -space, so $\mathcal{B}(\pi)$ cannot be constructed by the MPS-construction.*

By Lemma 5.3(4), the points of B on a $(p^e + 1)$ -secant form an \mathbb{F}_{p^e} -linear set of rank 2, i.e., a subline, which by field reduction corresponds to a regulus. We need the following information on the intersection of a regulus with a plane.

Lemma 5.6. *[11, Lemma 6, Corollary 13] A plane π meeting all elements of a regulus $\mathcal{B}(\ell)$, meets the point set of $\mathcal{B}(\ell)$ either in a line, or in two lines, or in a conic.*

Corollary 5.7. *Suppose that $\mathcal{B}(\ell)$ where ℓ is a line, is contained in $\mathcal{B}(K)$, where K is a blocking set with respect to lines in Π . Then the intersection of the point set of $\mathcal{B}(\ell)$ with a plane of Π contains a line.*

Proof. From Lemma 5.6, we get that the intersection of the point set of $\mathcal{B}(\ell)$ with a plane either contains a line or is a conic. But a conic does not block all lines in the plane π and $K \cap \pi$ is a blocking set with respect to lines in π , hence, this possibility does not occur. \square

Lemma 5.8. *A plane π meets the cone K , with vertex a $(t-2)$ -dimensional space Ω and base a small minimal blocking set \bar{B} in a plane Γ , skew from Ω , either in the plane π itself, in a number of lines through a fixed point or in a minimal blocking set equivalent to \bar{B} .*

Proof. Let π be a plane in $\Pi = \langle \Omega, \Gamma \rangle$. We have the following possibilities:

- π is contained in Ω . In this case, π is contained in K .
- π meets Ω in a line L . If π contains a point P of K , not on L , then $\langle \Omega, P \rangle$ is contained in K , hence, π is contained in K .
- π meets Ω in a point P . Since K is a blocking set with respect to lines in Π , K forms a blocking set with respect to lines in the plane π , so there exists a point of K in π , different from P . It is clear that every point of K in π , different from P , gives rise to a line contained in K through P .
- π is skew from Ω . Consider the mapping ϕ from Γ to π defined by mapping the point P of Γ to the intersection of the cone $\langle P, \Omega \rangle$ with π . It is clear that ϕ defines a collineation mapping \bar{B} onto the intersection of K with π , hence, K and π intersect in a minimal blocking set, equivalent to \bar{B} . \square

Lemma 5.9. *Let P be a point of a cone K with vertex a $(t-2)$ -dimensional space Ω and base a non-trivial minimal blocking set in a plane Γ skew from Ω and suppose that P is not contained in the vertex of K , then P lies on $(q^{t-1} - 1)/(q - 1)$ lines contained in K .*

Proof. Since the vertex Ω of the cone K is $(t-2)$ -dimensional, it is clear that a point P , not in Ω , lies on $(q^{t-1} - 1)/(q - 1)$ lines of $\langle P, \Omega \rangle$, which are contained in K . If P lies on another line L that is contained in K , then this line is skew from Ω and $\langle \Omega, L \rangle$ meets Γ in a line L' , contained in K , a contradiction since $K \cap \Gamma$ is a non-trivial minimal blocking set. \square

We now extend a property of small minimal planar blocking sets ([22, Proposition 4.17]) to blocking sets in $\text{PG}(n, q)$, $n \geq 2$.

Lemma 5.10. *If a small minimal blocking set B in $\text{PG}(n, p_0^h)$ has exponent e , $p_0 := p^e \geq 7$, p prime, then there exists a $(p_0 + 1)$ -secant to B .*

Proof. We proceed by induction, where the case $n = 2$ is Proposition 4.17 of [22]. Since B has exponent e , there is a hyperplane H with $|H \cap B| = 1 \pmod{p^e}$ and $|H \cap B| \neq 1 \pmod{p^{e+1}}$. It is clear that, since every subspace meets B in $1 \pmod{p^e}$ points and the number of points in B is equal to $1 \pmod{p^e}$, that we can find a line L in H with $\lambda = 1 \pmod{p^e}$ points and $\lambda \neq 1 \pmod{p^{e+1}}$. If a plane contains a point of B outside of L , then this plane contains at least p^{2e} points of B outside of L since the line through 2 points of B contains at least $p^e + 1$ points of B by Lemma 5.3(3). By Lemma 5.3(1), this implies that there exists a plane π through L without extra points of B . Let P be a point of π , not on L , then the projection B' of B from P onto a hyperplane through L and not through P is a small minimal blocking set in $\text{PG}(n-1, p_0^h)$ (see e.g. [23, Corollary 3.2]), which has, by our claim, exponent e . So by induction, we find a $(p_0 + 1)$ -secant to B' in H' . Since all points of B in π are on the line L , we have found a $(p_0 + 1)$ -secant to B . \square

Corollary 5.11. *Let B be a small minimal blocking set with exponent e in $\text{PG}(n, p_0^h)$, $p_0 := p^e \geq 7$, p prime, then there are at least $(p_0^{h-1} - 4p_0^{h-2} + 1)p_0 + 1$ points of B that each lie on at least $p_0^{h-1} - 4p_0^{h-2} + 1$ $(p_0 + 1)$ -secants to B .*

Proof. From Lemma 5.10, we get that there exists one $(p_0 + 1)$ -secant to B . Lemma 5.3 shows that through a point of this $(p_0 + 1)$ -secant there are at least $p_0^{h-1} - 4p_0^{h-2} + 1$ $(p_0 + 1)$ -secants. \square

As mentioned before, not all linear blocking sets are of Rédei-type, and we will now show that a small minimal blocking set arises from Construction 1 if and only if it is linear.

Theorem 5.12. *If $p \geq 7$, a small minimal blocking set with respect to the hyperplanes in $\text{PG}(n-1, q^t)$, $q = p^h$, p prime, with exponent e arises from Construction 1 in $\text{PG}(nht/e - 1, p^e)$ if and only if it is an \mathbb{F}_{p^e} -linear blocking set.*

Proof. Put $q_0 = p^e$, $q^t = q_0^{t_0}$ (hence $t_0 = ht/e$). Let B be an \mathbb{F}_{p^e} -linear blocking set, then $B = \mathcal{B}_{\mathcal{D}'}(\pi)$, where π is a t_0 -space in $\text{PG}(nt_0 - 1, q_0)$ and \mathcal{D}' is a Desarguesian $(t_0 - 1)$ -spread in $\text{PG}(nt_0 - 1, q_0)$. Let Ω be a $(t_0 - 2)$ -dimensional subspace of π and let Γ be a plane meeting π in a line \bar{B} , disjoint from Ω . It is clear that $B = \mathcal{B}_{\mathcal{D}'}(K)$, with K the

cone with vertex Ω and base \bar{B} ; note that \bar{B} is a minimal blocking set such that every point of \bar{B} lies on at least 2 tangent lines to \bar{B} . So this implies that B is obtainable from Construction 1.

So now assume that B is a small minimal blocking set obtained from Construction 1 for some $\Pi = \langle \Omega, \Gamma \rangle$ and \bar{B} where \bar{B} is a minimal blocking set in the plane Γ in $\text{PG}(nt_0 - 1, q_0)$, where Γ is skew from the $(t_0 - 2)$ -space Ω . We will show that $B = \mathcal{B}_{\mathcal{D}'}(\pi)$, where π is a t_0 -space in $\text{PG}(nt_0 - 1, q_0)$, where \mathcal{D}' is a Desarguesian $(t_0 - 1)$ -spread in $\text{PG}(nt_0 - 1, q_0)$. Corollary 5.11 states that there exists a point P of B on at least $q_0^{t_0-1} - 4q_0^{t_0-2} + 1$ $(q_0 + 1)$ -secants, and such that P is not an element of $\mathcal{B}(\Omega)$ since $|\mathcal{B}(\Omega)| \leq \frac{q_0^{t_0-1}-1}{q_0-1}$. Consider the spread element $\mathcal{B}_{\mathcal{D}'}(P)$ and its intersection S with the space $\Pi = \langle \Omega, \Gamma \rangle$. Let L_1, \dots, L_r be the $(q_0 + 1)$ -secants through P to B . Every L_i corresponds to a $(2t_0 - 1)$ -dimensional space in $\text{PG}(nt_0 - 1, q_0)$, denote by π_1, \dots, π_r the subspaces of Π that occur as the intersection of Π with the $(2t_0 - 1)$ -dimensional space corresponding to L_i . Note that two different spaces π_i and π_j meet exactly in the subspace S . From this, it follows that if S has dimension s , then the number r can be at most the number of spaces of dimension $s + 1$ through the s -dimensional subspace S in Π . This number equals $(q_0^{t_0-s} - 1)/(q_0 - 1)$, which is smaller than $q_0^{t_0-1} - 4q_0^{t_0-2} + 1$ if s is at least 1. So this implies that $s = 0$.

So $\mathcal{B}_{\mathcal{D}'}(P) \cap \langle \Omega, \Gamma \rangle$ is a point. By an easy counting, we find that there are more than $(q_0^{t_0-1} - 1)/(q_0 - 1)$ of the spaces π_i that are 1-or 2-dimensional. By Corollary 5.7 these all give rise to a line through P , contained in K . So by Lemma 5.9, we find that \bar{B} is a line which proves the statement. \square

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